The Spectra of Manhattan Street Networks

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Abstract

The multidimensional Manhattan street networks constitute a family of digraphs with many interesting properties, such as vertex symmetry (in fact they are Cayley digraphs), easy routing, Hamiltonicity, and modular structure. From the known structural properties of these digraphs, we determine their spectra, which always contain the spectra of hypercubes. In particular, in the standard (two-dimensional) case it is shown that their line digraph structure imposes the presence of the zero eigenvalue with a large multiplicity.

1 Introduction

The study of a class of directed torus networks known as Manhattan (street) networks has received significant attention since they were introduced independently and in different contexts by Morillo *et al.* [24] and Maxemchuk [22] as an unidirectional regular mesh structure locally resembling the topology of the avenues and streets of Manhattan or *l'Eixample* in downtown Barcelona; see Fig. 2.2. Morillo et al. [24] related the networks to plane tessellations, and this association facilitated the study of some main distance-related parameters, such as the distribution of the internodal distances, the diameter and the mean distance. In fact, most of the work on Manhattan street networks has been devoted to the computation of such parameters (for instance, the average distance is dealt with in [21]) and the generation of routing schemes for the 2-dimensional case [22]. Also, in this case, the study of spanning trees [11] in a Manhattan street network has allowed the computation of the diameter and the design of a multiport broadcasting algorithm. More recently, Varvarigos [27] evaluated the mean internodal distance, provided a shortest path routing algorithm, and also a decomposition into two edge-disjoint Hamiltonian cycles for the 2-dimensional case $N \times N$. Moreover, it has been shown that such networks admit optimal or quasi-optimal communication protocols, like the broadcasting (or dissemination of information from a given node to all the others) [11, 13].

The 3-dimensional natural extension of the Manhattan street networks has been considered by Banerjee *et al.*, see [2, 3], with the determination of the average distance of a 3-dimensional Manhattan street network, and a conjecture for higher dimensions. Chung and Agrawal [12] studied the diameter and provided routing schemes for a 3-dimensional construction based on a 2-dimensional Manhattan street networks, although the network obtained is not strictly a 3-dimensional Manhattan street network. The natural generalization to dimension $n \geq 2$ of Manhattan street networks, denoted M_n, has been recently studied by the first three authors of this paper [14]. In particular, it has been shown that M_n is a Cayley digraph of a subgroup of the n-dim version of the wallpaper group *pgg* (see for instance [16]).

Here we address the question of computing the spectrum of the n-dimensional Manhattan street networks, showing, among other things, that they contain the spectra of the hypercubes. The knowledge of the spectrum of a (di)graph is important for the estimation of relevant parameters, which are, in general, very hard to obtain by other methods. In particular, the spectrum of the adjacency matrix of a digraph contains information about its expansion properties, as it was first noticed by Tanner [26]. Similar connections have been established with other parameters such as the diameter [8, 9, 10, 25].

This paper is organized as follows. Section 2 is devoted to formally introduce the multidimensional Manhattan street network M_n and describe its main properties. Mainly, we give a useful alternative definition, based on the fact that M_n can be seen as a Cayley digraph related to a well-known cristallographic group. In the same section, it is also shown that the standard Manhattan street network M_2 has the structure of a line digraph, which is relevant to the study of its spectrum. The main body of this paper is in Section 3, where the spectral properties (eigenvalues and eigenvectors) of the two-dimensional case are fully characterized, and the computation complexity of the multidimensional case is drastically reduced.

2 The Multidimensional Manhattan Street Network

In this section we recall the definition and basic properties of the networks under study. With this aim, we begin with some background on digraphs and their spectra.

2.1 Preliminaries

We model networks using digraphs. A directed graph or *digraph* for short, denoted $G = (V, E)$, consists of a set of *vertices* V, together with a set of *arcs* A, which can be seen as ordered pairs of vertices, $A \subset V \times V = \{(u, v) : u, v \in V\}$. An arc (u, v) is usually depicted as an arrow with *initial* vertex u and *terminal* vertex v, that is, $u \to v$. The *indegree* $\delta^-(u)$ (respectively, *outdegree* $\delta^+(u)$) of a vertex u is the number of arcs with initial (respectively, terminal) vertex u. Then G is δ*-regular* when $\delta^-(u) = \delta^+(u) = \delta$ for every vertex $u \in V$.

A *homomorphism* φ from a digraph $G = (V, A)$ to a digraph $H = (V', A')$ is a mapping from V to V' preserving adjacencies, that is, $(u, v) \in A$ if and only if $(\varphi(u), \varphi(v)) \in A'$.

Recall also that a partition $\pi = (C_1, \ldots, C_k)$ of the vertex set V is *equitable* or *regular* if, for all i and j, the number of neighbors that a vertex in C_i has in C_j only depends on i and j, see [19].

The standard definitions and basic results about graphs and digraphs not defined here can be found in [4, 7].

Let us now recall a useful result from spectral graph theory. For any digraph, it is known that the components of its eigenvalues can be seen as charges on each vertex (see [18, 19]). More precisely, suppose that $G = (V, A)$ is a digraph with adjacency matrix $\boldsymbol{A}$ and λ-eigenvector $\boldsymbol{v}$. Then the charge of a vertex $i \in V$ is the corresponding entry v_i of $\boldsymbol{v}$, and the equation $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ means that

$$\sum_{i \to j} v_j = \lambda v_i \quad \text{for every} \quad i \in V. \tag{1}$$

That is, each vertex "absorbs" the charges of its out-neighbors to get a final charge λ times the one it had originally.

2.2 Definition and structure

As mentioned in the Introduction, the standard *Manhattan street network* was defined as a 2-regular digraph in the following way: every vertex is represented by a pair of integers $\boldsymbol{u} = (u_1, u_2)$, with $0 \le u_i \le N_i - 1$, for some even integers N_i, $i = 1, 2$, and each vertex $\boldsymbol{u}$ has two outgoing arcs: one horizontal $(u_1 \pm 1, u_2)$ and the other vertical $(u_1, u_2 \pm 1)$ (where the sign depends on the parity of the other component and the arithmetic must be understood modulo N_i). More precisely, a horizontal arc points to *est* (respectively, *west*) if it is on an *even* (respectively, *odd*) row. Similarly, a vertical arc points to *north* (respectively, *south*) if it is on an *even* (respectively, *odd*) column. Locally the structure obtained is as shown in Fig. 2.2, and it corresponds to a standard pattern for the allowed traffic directions in some neighborhoods of our modern cities, like New York or Barcelona, with their system of straight orthogonal streets.

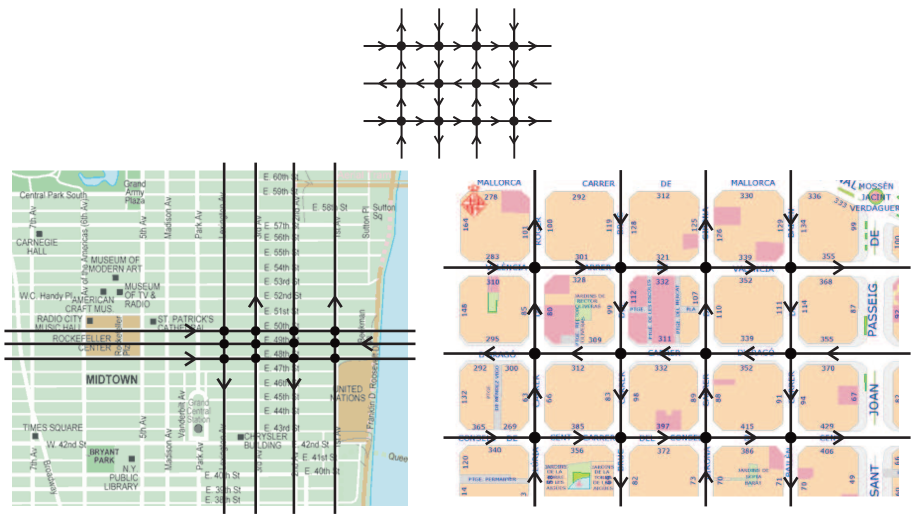


Figure 1: The local pattern of a Manhattan street network and two real-life examples: Orthogonal streets of Manhattan and l'Eixample in Barcelona.

The natural extension of Manhattan street networks to higher dimensions was formally introduced in [14]. Its standard definition goes as follows:

Definition 2.1. *Given n even positive integers $N_1, N_2, \ldots, N_n$, the n-dimensional Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$ is a digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$. Thus, each of its vertices is represented by an n-vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, with $0 \le u_i \le N_i - 1$, $i = 1, 2, \ldots, n$. The arc set $A(M_n)$ is defined by the following adjacencies:*

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \rightarrow \quad (u_1, \ldots, u_i + (-1)^{\sum_{j \ne i} u_j}, \ldots, u_n) \qquad (1 \le i \le n). \tag{2}$$

Therefore, M_n is an n-regular digraph on $N = \prod_{i=1}^{n} N_i$ vertices. In particular, when $N_i = 2$, $1 \le i \le n$, the n-dimensional Manhattan street network is isomorphic to the symmetric digraph Q_n^*, with Q_n being the hypercube of dimension n or *n-cube*.

Some other simple consequences of the definition of M_n follow (see [14]).

Lemma 2.2. *Every n-dimensional Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$ satisfies the following properties:*

(a) *Given any permutation* π *of the numbers* $N_1, N_2, \ldots, N_n$*, say* $P_1, P_2, \ldots, P_n$*, the Manhattan street networks* M_n *and* $M_n^\pi = M(P_1, P_2, \ldots, P_n)$ *are isomorphic digraphs.*

(b) M_n *is isomorphic to its converse* $\overline{M}_n$*.*

(c) *For any fixed* $n-k$ *integers* $x_i \in \mathbb{Z}_{N_i}$*,* $i = k+1, k+2, \ldots, n$*, the subdigraph of* M_n *induced by the vertices of the form* $(u_1, u_2, \ldots, u_k, x_{k+1}, \ldots, x_n)$ *is either the* k*-dimensional Manhattan street networks* $M_k = M(N_1, N_2, \ldots, N_k)$ *or its converse* $\overline{M}_k$*, depending on whether* $\alpha := \sum_{i=k+1}^{n} x_n$ *is even or odd, respectively.*

(d) M_n *is both a* 2^n*-partite and bipartite digraph.*

(e) *There exists an homomorphism from* M_n *to the symmetric hypercube* Q_n^**.*

For instance, (d) holds since M_n has independent sets $V_{\boldsymbol{b}}$, where $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ is a binary n-string. A vertex $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ belongs to $V_{\boldsymbol{b}}$ when the parities of u_i and b_i coincide for every $1 \le i \le n$. In particular, M_n is bipartite with stable vertex sets V_0 and V_1 constituted by the vertices whose corresponding binary string has, respectively, even or odd *Hamming weight*, that is, number of 1's.

2.3 An alternative definition

In [14], the authors proved some structural results concerning the symmetries of the Manhattan street networks which lead to the following useful alternative presentation.

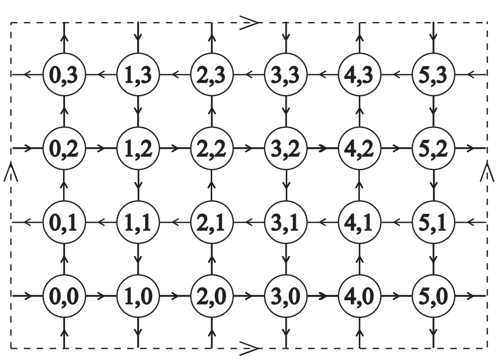
Definition 2.3. *The vertex set of* $M_n = M(N_1, N_2, \ldots, N_n)$ *is, as before,* $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$ *and the* $(i\text{-})$*arcs are now:*

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \rightarrow \quad (-u_1, \ldots, -u_{i-1}, u_i + 1, -u_{i+1}, \ldots, -u_n) \qquad (1 \le i \le n). \tag{3}$$

In fact, the (involutive) isomorphism Ψ from the standard definition to the new presentation is defined by:

$$\Psi(u_1, u_2, \ldots, u_n) = ((-1)^{\sum_{j\neq 1} u_j} u_1, (-1)^{\sum_{j\neq 2} u_j} u_2, \ldots, (-1)^{\sum_{j\neq n} u_j} u_n). \tag{4}$$

As an example, Fig. 2 shows both, the standard definition and the new presentation of $M(6,4)$. (Directed dashed lines represent the identification of parallel sides of the rectangle corresponding to the torus surface.)



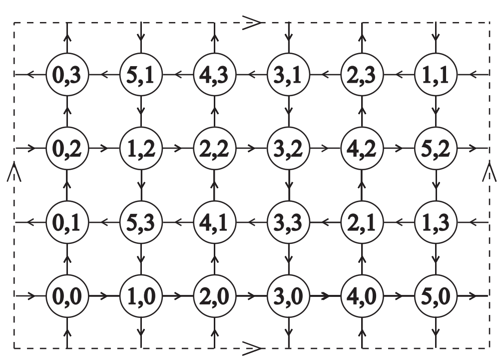


Figure 2: The Manhattan street network $M(6,4)$ with vertices labeled according to the standard (on the left) and alternative (on the right) definitions.

2.4 The line digraph structure

Recall that, given a digraph $G = (V, A)$ with n vertices and m arcs, its line digraph $LG = (V_L, A_L)$ has vertices representing the arcs of G, so that we identify each vertex $ij \in V_L$ with the arc $(i, j) \in A$; and its adjacencies are naturally induced by the arc adjacencies in G. More precisely, vertex $ij \in V_L$ is adjacent to vertex jk since the arc $(i, j) \in A$ has the same terminal vertex as the initial vertex of (j, k). Thus, the order of LG equals the size m of G and, if G is δ-regular, so is LG and it has δn arcs. Also, it is known that if G is different from a (directed) cycle and has diameter D, then its line digraph LG has diameter $D + 1$. Some interesting properties of line digraphs can be found in [15, 17]. Among them, the spectrum of the line digraph LG has the same non-zero eigenvalues as G, including (algebraic) multiplicities. In fact, the eigenvalue sets only differ in the number of zeros since their corresponding characteristic polynomials, p_{LG} and p_G, satisfy (see [1, 23]):

$$p_{LG}(x) = x^{m-n} p_G(x).$$

The next result shows that the line digraph structure is inherent to the 2-dimensional case.

Lemma 2.4. *For any N_1, N_2, the 2-dimensional Manhattan street network M_2 is a line digraph.*

Proof. It suffices to check the *Heuchenne's condition* [20], which says that a digraph is a line digraph if and only if it has no multiple arc and the out-neighbor (or in-neighbor) sets of any two of its vertices are either identical or disjoint. If two vertices $(i, j), (i', j') \in A$ have a common neighbor either

(a) $\quad i + 1 = i' + 1$ and $\ -j = -j'$,

or

(b) $\quad i + 1 = -i'$ and $\ -j = j' + 1$.

The first situation leads to identical vertices $(i, j) = (i', j')$. The second assumption gives $i' + 1 = -i$ and $-j' = j + 1$. In this case, vertices (i, j) and (i', j') also have common neighborhood (see Fig. 3). In fact, notice that $\Gamma^+(i, j) = \Gamma^+(i', j')$ iff $i + i' = j + j' = -1$.
□

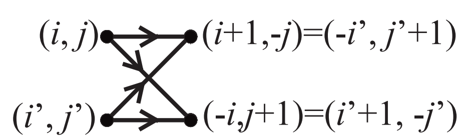


Figure 3: The Heuchenne's condition in M_2.

In fact, for $n \geq 3$, the n-dimensional Manhattan street networks are never line digraphs, as they do not fulfill Heuchenne's condition. For example, in $M(8, 6, 10)$ the vertices $(1, 1, 5)$ and $(6, 1, 4)$ are both adjacent to vertices $(2, 5, 5)$ and $(7, 5, 6)$ but they do not have identical out-neighborhood

$$\Gamma^+(1, 1, 5) = \{(2, 5, 5), (7, 2, 5), (7, 5, 6)\} \neq \{(7, 5, 6), (2, 2, 6), (2, 5, 5)\} = \Gamma^+(6, 1, 4).$$

A direct consequence of Lemma 2.4 and the results in [18], which is relevant to our study, is that the spectrum of $M(N_1, N_2)$ has the eigenvalue 0 with geometric multiplicity at least $N_1N_2/2$. For instance, Fig. 4 illustrates a 0-eigenvector of $M_2(6, 4)$ as a distribution of charges ± 1 and 0's on its vertices. Indeed note that, using (1), every vertex has out-neighbors whose total charge add up to 0. Analogously, each of the remaining $N_1N_2/2-1=11$ (linearly independent) 0-eigenvectors would be obtained by putting ± 1 on the two vertices of a given dotted diagonal and 0 elsewhere.

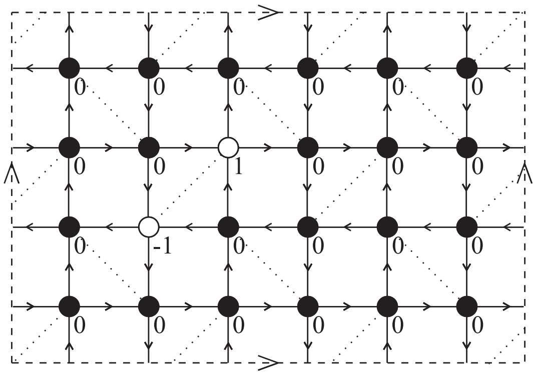


Figure 4: The 0-eigenvectors of $M_2(6,4)$.

3 The spectrum

We first recall some results concerning the spectra of the (directed) cycle C_N, the direct product of two cycles $C_{N_1}\square C_{N_2}$ and the n-cube Q_n.

The eigenvalues of the cycle C_N are the N roots of the unity, $\omega^k = e^{i\frac{2\pi}{N}k}$, $0\le k\le N-1$, and $\boldsymbol{\phi}^k=(1,\omega^k,\omega^{2k},\dots,\omega^{(N-1)k})$ is an eigenvector of ω^k. Similarly, the eigenvalues of $C_{N_1}\square C_{N_2}$ are $\omega^k+\tau^l=e^{i\frac{2\pi}{N_1}k}+e^{i\frac{2\pi}{N_2}l}$, $0\le k\le N_1-1$, $0\le l\le N_2-1$, whose respective eigenvectors $\boldsymbol{\phi}=\boldsymbol{\phi}^{(k,l)}$ have components $\phi_{(i,j)}=\omega^{ik}\tau^{jl}$. Here it is worth noting that the vector sets $\{\boldsymbol{\phi}^k\}_k$ and $\{\boldsymbol{\phi}^{(k,l)}\}_{k,l}$ are the orthogonal bases involved in the computation of the (respectively, 1-dim and 2-dim) discrete Fourier transforms (DFT's) [6].

Moreover, the spectrum of the n-cube Q_n is

$$\operatorname{sp} Q_n=\{(n-2k)^{\binom{n}{k}} : k=0,1,\dots,n\},$$

where the superscripts denote multiplicities. In this case, the corresponding eigenvectors have entries ± 1 and are defined below (as $M_n(2,2,\dots,2)\cong Q_n$); see for instance [5].

We begin our study with a general result concerning the n-dimensional case. So, the following proposition shows that the spectrum of an n-dimensional Manhattan street network contains the spectrum of the n-cube Q_n.

Proposition 3.1. *The spectrum of the n-dimensional Manhattan street network $M_n=M(N_1,N_2,\dots,N_n)$ contains all the eigenvalues (including multiplicities) of the n-cube Q_n:*

$$\operatorname{sp} Q_n\subseteq \operatorname{sp} M_n,$$

with equality when $N_i=2$, $1\le i\le n$.

Furthermore, for every subset $I_k\subset\{1,2,\dots,n\}$ of cardinality k, $0\le k\le n$, the vector $\boldsymbol{w}$ with components $w_{\boldsymbol{u}}=\prod_{i\in I_k}(-1)^{u_i}$, where $\boldsymbol{u}=(u_1,u_2,\dots,u_n)$ is an eigenvector for the eigenvalue $\lambda=n-2k$.

Proof. The independent sets $V_{\boldsymbol{b}}$, defined after Lemma 2.2, constitute an equitable partition π of the set of vertices of M_n. The corresponding quotient digraph M_n/π is clearly isomorphic to the n-cube Q_n. Thus, the characteristic polynomial of Q_n divides the characteristic polynomial of M_n, see [19, pag. 78]. Moreover, when $N_i = 2$ for every $1 \le i \le n$, we have equality since $M_n(2, 2, \ldots, 2) \cong Q_n$.

In order to find the eigenvectors associated to the eigenvalues, we think of each component of a λ-eigenvector $\boldsymbol{w}$ as a "charge" in the corresponding vertex. Then the sum of the charges of the out-neighbors of vertex $\boldsymbol{u}$ is λ times the charge of vertex $\boldsymbol{u}$, see [18]. If the charge of vertex $\boldsymbol{u}$ is $w_{\boldsymbol{u}} = \prod_{i \in I_k}(-1)^{u_i}$, then among its n out-neighbors, there are $(n-k)$ vertices with charge $w_{\boldsymbol{u}}$ (namely, those adjacent from $\boldsymbol{u}$ through i-arcs, $i \notin I_k$) and k vertices with charge $-w_{\boldsymbol{u}}$ (those adjacent from $\boldsymbol{u}$ through i-arcs, $i \in I_k$). Thus,

$$\lambda w_{\boldsymbol{u}} = (n-k)w_{\boldsymbol{u}} - k w_{\boldsymbol{u}} = (n-2k)w_{\boldsymbol{u}}.$$

To complete the proof, notice that, for every value of k, the $\binom{n}{k}$ possible choices of I_k give rise to the same number of independent eigenvectors $\boldsymbol{w}$. Indeed, observe that the weight $w_{\boldsymbol{u}}$ of vertex $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ only depends on the parity of entries in I_k. Thus, the vectors $\boldsymbol{w}$ are also the eigenvectors of the n-cube $Q_n \cong M_n(2, 2, \ldots, 2)$. Consequently, the geometric multiplicity of the eigenvalue $\lambda = n - 2k$ is as required. $\square$

3.1 The two-dimensional case

The next result shows how the eigenvalues of the 2-dimensional Manhattan street network $M(N_1, N_2)$ can be computed in terms of the eigenvalues of the directed cycles C_{N_i}, $i = 1, 2$.

Theorem 3.2. *The eigenvalues of the 2-dimensional Manhattan network $M_2 = M(N_1, N_2)$ are*

$$0, \pm\sqrt{2\cos\left(\tfrac{4\pi k}{N_1}\right) + 2\cos\left(\tfrac{4\pi l}{N_2}\right)} \quad \textit{for} \quad 0 \le k \le \tfrac{N_1}{2} - 1,\ 0 \le l \le \tfrac{N_2}{2} - 1. \tag{5}$$

Moreover, the geometric multiplicity of every non-zero eigenvalue coincides with the times it appears in (5)*, whereas the geometric multiplicity of the eigenvalue zero satisfies $m(0) \ge (N_1N_2)/2$, and equality happens when $N_i \not\equiv 0 \pmod 4$, $i = 1, 2$.*

Proof. Let $\boldsymbol{A}$ be the adjacency matrix of the 2-dimensional Manhattan network $M_2 = M(N_1, N_2)$. Let $\lambda_1 = e^{i\frac{2\pi}{N_1}k}$ and $\lambda_2 = e^{i\frac{2\pi}{N_2}l}$ be eigenvalues of the cycles C_{N_1} and C_{N_2}, for some $0 \le k \le N_1 - 1$ and $0 \le l \le N_2 - 1$, respectively. Then, from their corresponding eigenvectors

$$\begin{aligned} \boldsymbol{x} &= (x_0, x_1, \ldots, x_{N_1-1}) = (1, \lambda_1, \lambda_1^2, \ldots, \lambda_1^{N_1-1}), \\ \boldsymbol{y} &= (y_0, y_1, \ldots, y_{N_2-1}) = (1, \lambda_2, \lambda_2^2, \ldots, \lambda_2^{N_2-1}). \end{aligned}$$

we define a λ-eigenvector $\boldsymbol{w}$ in M_2, whose components are of the form:

$$w_{(i,j)} = \alpha x_i y_j + \beta x_{-i} y_{-j} + \gamma x_i y_{-j} + \delta x_{-i} y_j, \tag{6}$$

for some constants $\alpha, \beta, \gamma, \delta$ to be determined. Notice that, in terms of the eigenvectors of $C_{N_1} \square C_{N_2}$, this corresponds to take the vector

$$\boldsymbol{w} = \boldsymbol{w}^{(k,l)} = \alpha \boldsymbol{\phi}^{(k,l)} + \beta \boldsymbol{\phi}^{(-k,-l)} + \gamma \boldsymbol{\phi}^{(k,-l)} + \delta \boldsymbol{\phi}^{(-k,l)}. \tag{7}$$

Thus, in constructing $\boldsymbol{w}$, we are not only using $\boldsymbol{x}$ and $\boldsymbol{y}$, but also their conjugate vectors $\overline{\boldsymbol{x}} = (\overline{x_0}, \overline{x_1}, \ldots, \overline{x_{N_1-1}})$ and $\overline{\boldsymbol{y}} = (\overline{y_0}, \overline{y_1}, \ldots, \overline{y_{N_2-1}})$, which correspond to the eigenvalues $\overline{\lambda_1} = e^{-i\frac{2\pi}{N_1}k}$ and $\overline{\lambda_2} = e^{-i\frac{2\pi}{N_2}l}$, respectively. Therefore, without loss of generality, we can restrict ourselves to the ranges $0 \le k \le N_1/2$ and $0 \le l \le N_2/2$.

Now, interpreting again the entries of an eigenvector as charges in each vertex—see (1)—the following equalities hold:

$$\begin{aligned} x_{i+1} &= \lambda_1 x_i, \quad x_{-i} = \lambda_1 x_{-i-1}, \quad i \in \mathbb{Z}_{N_1}; \\ y_{j+1} &= \lambda_2 y_j, \quad y_{-j} = \lambda_2 y_{-j-1}, \quad j \in \mathbb{Z}_{N_2}. \end{aligned}$$

Besides, as $\lambda_i^{-1} = \overline{\lambda_i}$, the two rightmost equalities can be written as:

$$\begin{aligned} x_{-i-1} &= \overline{\lambda_1} x_{-i}, \qquad i \in \mathbb{Z}_{N_1}; \\ y_{-j-1} &= \overline{\lambda_2} y_{-j}, \qquad j \in \mathbb{Z}_{N_2}. \end{aligned}$$

Also by (1), we have that, for every vertex (i,j) of M_2,

$$(\boldsymbol{A}\boldsymbol{w})_{(i,j)} = \sum_{(i',j')\leftarrow(i,j)} w_{(i',j')} = \lambda\, w_{(i,j)}.$$

Thus, taking into account that vertex (i,j) is adjacent to the vertices $(i+1,-j)$ and $(-i,j+1)$, and using the expression of $w_{(i,j)}$ in (6), the above equation becomes:

$$\begin{aligned} w_{(i+1,-j)} + w_{(-i,j+1)} &= \alpha x_{i+1} y_{-j} + \beta x_{-i-1} y_j + \gamma x_{i+1} y_j + \delta x_{-i-1} y_{-j} \\ &+ \alpha x_{-i} y_{j+1} + \beta x_i y_{-j-1} + \gamma x_{-i} y_{-j-1} + \delta x_i y_{j+1} \\ &= \lambda(\alpha x_i y_j + \beta x_{-i} y_{-j} + \gamma x_i y_{-j} + \delta x_{-i} y_j), \end{aligned}$$

whence, for every i, j,

$$\begin{aligned} & \alpha\lambda_1 x_i y_{-j} + \frac{\beta}{\lambda_1} x_{-i} y_j + \gamma\lambda_1 x_i y_j + \frac{\delta}{\lambda_1} x_{-i} y_{-j} \\ + \quad & \alpha\lambda_2 x_{-i} y_j + \frac{\beta}{\lambda_2} x_i y_{-j} + \frac{\gamma}{\lambda_2} x_{-i} y_{-j} + \delta\lambda_2 x_i y_j \\ = \quad & (\gamma\lambda_1 + \delta\lambda_2) x_i y_j + \left(\frac{\delta}{\lambda_1} + \frac{\gamma}{\lambda_2}\right) x_{-i} y_{-j} + \left(\alpha\lambda_1 + \frac{\beta}{\lambda_2}\right) x_i y_{-j} + \left(\frac{\beta}{\lambda_1} + \alpha\lambda_2\right) x_{-i} y_j \\ = \quad & \lambda(\alpha x_i y_j + \beta x_{-i} y_{-j} + \gamma x_i y_{-j} + \delta x_{-i} y_j) \end{aligned}$$

or, in terms of the corresponding vectors,

$$\begin{aligned} & (\gamma\lambda_1 + \delta\lambda_2)\boldsymbol{\phi}^{(k,l)} + \left(\frac{\delta}{\lambda_1} + \frac{\gamma}{\lambda_2}\right)\boldsymbol{\phi}^{(-k,-l)} + \left(\alpha\lambda_1 + \frac{\beta}{\lambda_2}\right)\boldsymbol{\phi}^{(k,-l)} + \left(\frac{\beta}{\lambda_1} + \alpha\lambda_2\right)\boldsymbol{\phi}^{(-k,l)} \\ = \quad & \lambda\alpha\boldsymbol{\phi}^{(k,l)} + \lambda\beta\boldsymbol{\phi}^{(-k,-l)} + \lambda\gamma\boldsymbol{\phi}^{(k,-l)} + \lambda\delta\boldsymbol{\phi}^{(-k,l)}. \end{aligned} \tag{8}$$

Now we must distinguish four cases, depending on the values of k and l:

(a) Let us first assume that $0 < k < N_1/2$ and $0 < l < N_2/2$. Then, since $k \neq -k$ and $l \neq -l$, the four vectors $\boldsymbol{\phi}^{(\pm k, \pm l)}$ are linearly independent and their respective coefficients must be equal. Thus, (8) yields the matricial equation:

$$\begin{pmatrix} 0 & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \frac{1}{\lambda_2} & \frac{1}{\lambda_1} \\ \lambda_1 & \frac{1}{\lambda_2} & 0 & 0 \\ \lambda_2 & \frac{1}{\lambda_1} & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{9}$$

Consequently, λ is an eigenvalue of the above matrix, whose characteristic polynomial is

$$p(x) = x^4 - x^2(\lambda_1^2 + \lambda_1^{-2} + \lambda_2^2 + \lambda_2^{-2}), \tag{10}$$

with zeros $x = 0$ (double) and $x = \pm\sqrt{\lambda_1^2+\lambda_1^{-2}+\lambda_2^2+\lambda_2^{-2}}$, whereas the corresponding eigenvectors are the rows of the matrix

$$\boldsymbol{B} = \begin{pmatrix} 1 & -\lambda_1\lambda_2 & 0 & 0 \\ 0 & 0 & -\overline{\lambda_1}\lambda_2 & 1 \\ \frac{\lambda\lambda_1^2\lambda_2}{1+\lambda_1^2\lambda_2^2} & \frac{\lambda\lambda_1}{1+\lambda_1^2\lambda_2^2} & \lambda_1\overline{\lambda_2} & 1 \\ -\frac{\lambda\lambda_1^2\lambda_2}{1+\lambda_1^2\lambda_2^2} & -\frac{\lambda\lambda_1}{1+\lambda_1^2\lambda_2^2} & \lambda_1\overline{\lambda_2} & 1 \end{pmatrix}, \tag{11}$$

where

$$\lambda = \sqrt{\lambda_1^2+\lambda_1^{-2}+\lambda_2^2+\lambda_2^{-2}} = \sqrt{2\cos\left(\frac{4\pi k}{N_1}\right)+2\cos\left(\frac{4\pi l}{N_2}\right)}. \tag{12}$$

Thus, for each pair of eigenvalues λ_1 and λ_2 of the cycles C_{N_1}, C_{N_2}, the eigenvectors of the eigenvalues 0 (with multiplicity two) and $\pm\lambda$ of $M_2(N_1,N_2)$ are, respectively,

$$\begin{aligned}
\boldsymbol{w}_1^{(k,l)} &= \boldsymbol{\phi}^{(k,l)} - \lambda_1\lambda_2\boldsymbol{\phi}^{(-k,-l)}, \\
\boldsymbol{w}_2^{(k,l)} &= -\overline{\lambda_1}\lambda_2\boldsymbol{\phi}^{(k,-l)} + \boldsymbol{\phi}^{(-k,l)}, \\
\boldsymbol{w}_3^{(k,l)} &= \tfrac{\lambda\lambda_1^2\lambda_2}{1+\lambda_1^2\lambda_2^2}\boldsymbol{\phi}^{(k,l)} + \tfrac{\lambda\lambda_1}{1+\lambda_1^2\lambda_2^2}\boldsymbol{\phi}^{(-k,-l)} + \lambda_1\overline{\lambda_2}\boldsymbol{\phi}^{(k,-l)} + \boldsymbol{\phi}^{(-k,l)}, \\
\boldsymbol{w}_4^{(k,l)} &= -\tfrac{\lambda\lambda_1^2\lambda_2}{1+\lambda_1^2\lambda_2^2}\boldsymbol{\phi}^{(k,l)} - \tfrac{\lambda\lambda_1}{1+\lambda_1^2\lambda_2^2}\boldsymbol{\phi}^{(-k,-l)} + \lambda_1\overline{\lambda_2}\boldsymbol{\phi}^{(k,-l)} + \boldsymbol{\phi}^{(-k,l)}.
\end{aligned} \tag{13}$$

First, note that, as the four vectors $\boldsymbol{\phi}^{(\pm k,\pm l)}$ are linearly independent (in fact orthogonal), the two eigenvectors $\boldsymbol{w}_1^{(k,l)}$, $\boldsymbol{w}_2^{(k,l)}$, corresponding to the eigenvalue 0, also are. Thus, the zero eigenvalue has geometric multiplicity $2(N_1/2-1)(N_2/2-1)$.

Moreover, when the value of λ in (12) is not zero, all the eigenvectors $\boldsymbol{w}_i^{(k,l)}$, $1 \le i \le 4$, are also linearly independent. This is because of the linear independence of the vectors $\boldsymbol{\phi}^{(\pm k,\pm l)}$ and the fact that the determinant of the matrix in (11) is

$$\det \boldsymbol{B} = \frac{2\lambda(\lambda_1^2+\lambda_2^2)}{\lambda_2}.$$

Thus, when $\lambda \neq 0$, it has linearly independent rows.

Let us now see that $\lambda = 0$ when $N_i \equiv 0 \pmod 4$, for some $i = 1,2$. Indeed, from (12), $\lambda = 0$ iff $\cos(\frac{4\pi k}{N_1}) + \cos(\frac{4\pi l}{N_2}) = 0$, that is, either, both angles are equal to an odd multiple of $\frac{\pi}{2}$ or their sum or difference is also an odd multiple of $\frac{\pi}{2}$.

In the first case, we have $\lambda_1^2+\lambda_2^2 = 0$ and $N_1 \equiv N_2 \equiv 0 \pmod 4$. In the second case, $\lambda_i^4+1 = 0$ and $N_i \equiv 0 \pmod 4$ for at least one value of $i \in \{1,2\}$.

Finally, notice also that, if $(k',l') \neq (k,l)$, every vector $\boldsymbol{w}_i^{(k',l')}$ is orthogonal to every vector $\boldsymbol{w}_i^{(k,l)}$, $1 \le i \le 4$.

Summarizing, in case (a) we have:

- A zero eigenvalue with geometric multiplicity $2(N_1/2-1)(N_2/2-1)$.
- If $N_i \not\equiv 0 \pmod 4$, $i = 1,2$, there are $2(N_1/2-1)(N_2/2-1)$ non-zero eigenvalues whose sum of geometric multiplicities add up to the same amount.

(b) $0 < k < \frac{N_1}{2}$, $l \in \{0, \frac{N_2}{2}\}$; that is, $k \neq -k$, $\lambda_2 = \pm 1$. The expression of the vector in (7) is now

$$\boldsymbol{w} = \alpha\boldsymbol{\phi}^{(k,l)} + \delta\boldsymbol{\phi}^{(-k,l)},$$

which corresponds to take $\beta = \gamma = 0$ in equation (8). In matricial form, we obtain two equations:

$$\begin{pmatrix} \lambda_1 & \pm 1 \\ \pm 1 & \frac{1}{\lambda_1} \end{pmatrix} \begin{pmatrix} \alpha \\ \delta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \delta \end{pmatrix}, \tag{14}$$

taking either the plus signs or the negative signs. Both matrices have the same characteristic polynomial

$$p(x) = x(x - \lambda_1 - \tfrac{1}{\lambda_1}). \tag{15}$$

Then, altogether we have the eigenvalue 0 and

$$\lambda = \lambda_1 + \overline{\lambda_1} = 2\cos\left(\tfrac{2\pi k}{N_1}\right) = \sqrt{2\cos\left(\tfrac{4\pi k}{N_1}\right) + 2}. \tag{16}$$

Note that the above corresponds to take $l = \pm 1$ in (12).

So, in this case we have the eigenvalue 0 with geometric multiplicity $2\left(\frac{N_1}{2} - 1\right)$ and equal number of different λ eigenvalues from (16).

(c) $k \in \{0, \frac{N_1}{2}\}$, $0 < l < \frac{N_2}{2}$; that is, $\lambda_1 = \pm 1$, $l \neq -l$. This is similar to the above case with the eigenvalues being 0 and

$$\lambda = \lambda_2 + \overline{\lambda_2} = 2\cos\left(\tfrac{2\pi l}{N_2}\right) = \sqrt{2\cos\left(\tfrac{4\pi l}{N_2}\right) + 2}. \tag{17}$$

Again, the geometric multiplicity of 0 is $2\left(\frac{N_2}{2} - 1\right)$, and there are also $2\left(\frac{N_2}{2} - 1\right)$ eigenvalues from (17).

(d) $k \in \{0, \frac{N_1}{2}\}$, $l \in \{0, \frac{N_2}{2}\}$; that is, $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$. As $k = -k$ and $l = -l$, the expression of the vector (7) is now $\boldsymbol{w} = \alpha \boldsymbol{\phi}^{(k,l)}$. This corresponds to take $\beta = \gamma = \delta = 0$ in (7). Thus, equation (8) becomes

$$\begin{array}{rcll} 2\alpha\boldsymbol{\phi}^{(0,0)} & = & \lambda\alpha\boldsymbol{\phi}^{(0,0)} & \text{for } \lambda_1 = \lambda_2 = 1, \\ 0 & = & \lambda\alpha\boldsymbol{\phi}^{(0,N_2/2)} & \text{for } \lambda_1 = 1, \lambda_2 = -1, \\ 0 & = & \lambda\alpha\boldsymbol{\phi}^{(N_1/2,0)} & \text{for } \lambda_1 = -1, \lambda_2 = 1, \\ -2\alpha\boldsymbol{\phi}^{(N_1/2,N_2/2)} & = & \lambda\alpha\boldsymbol{\phi}^{(N_1/2,N_2/2)} & \text{for } \lambda_1 = \lambda_2 = -1, \end{array}$$

which gives, respectively, the eigenvalues $\lambda = 2$, $\lambda = 0$ (double) and $\lambda = -2$.

Finally, the sum of the non-zero different eigenvalues from the above discussion, is $\frac{N_1}{2} + \frac{N_2}{2} + 2$ and the same amount for $\lambda = 0$. Thus, together with the eigenvalues obtained when $k \neq -k, l \neq -l$, we conclude that the multiplicity of the zero eigenvalue is, at least, $\frac{N_1 N_2}{2}$ as stated. □

Corollary 3.3. *The spectrum of the 2-dimensional Manhattan street network* $M_2 = M(N_1, N_2)$, *with* $N_1, N_2 \not\equiv 0 \pmod 4$, *is*

$$\operatorname{sp} M_2 = \left\{ 0^{\frac{N_1 N_2}{2}}, \pm\sqrt{2\cos\left(\tfrac{4\pi k}{N_1}\right) + 2\cos\left(\tfrac{4\pi l}{N_2}\right)} \;\middle|\; 0 \leq k < \tfrac{N_1}{2}, 0 \leq l < \tfrac{N_2}{2} \right\},$$

where the superscript denotes multiplicity. Moreover, the algebraic and geometric multiplicities coincide.

Notice that, in the above results, we can also use that a λ-eigenvector of the cycle C_N is $\boldsymbol{\omega}=(1,\lambda,\lambda^2,\ldots,\lambda^{N-1})$, although their presentation is not simplified.

Some examples of the spectra obtained from the result given in (5) are the following:

$$\begin{aligned}
\operatorname{sp} M(2,2) &= \{0^2, \pm 2\},\\
\operatorname{sp} M(4,2) &= \{0^6, \pm 2\},\\
\operatorname{sp} M(4,4) &= \{0^{12}, \pm 2, \pm 2\,i\},\\
\operatorname{sp} M(4,6) &= \{0^{14}, \pm 2, (\pm 1)^2, (\pm\sqrt{3}\,i)^2\},\\
\operatorname{sp} M(8,4) &= \{0^{20}, \pm 2, (\pm\sqrt{2})^2, (\pm\sqrt{2}i)^2, \pm 2\,i\},\\
\operatorname{sp} M(6,6) &= \{0^{18}, \pm 2, (\pm 1)^4, (\pm\sqrt{2}i)^4\},\\
\operatorname{sp} M(8,6) &= \{0^{26}, (\pm 1)^2, \pm 2, (\pm i)^4, (\pm\sqrt{3}\,i)^2, (\pm\sqrt{2})^2\},\\
\operatorname{sp} M(8,8) &= \{0^{46}, \pm 2, (\pm\sqrt{2})^4, (\pm\sqrt{2}\,i)^4, \pm 2\,i\},\\
\operatorname{sp} M(6,10) &= \{0^{30}, \pm 2, (\pm 1)^2, (\pm 2\cos(\pi/5))^2, (\pm 2\cos(2\pi/5)\,)^2,\\
&\quad (\pm i\,\sqrt{3-4\cos^2(\pi/5)}\,)^4, (\pm i\,\sqrt{3-4\cos^2(2\pi/5)})^4\},
\end{aligned}$$

in particular for $M(2,2), M(6,6), M(6,10)$ the given multiplicities are geometric.

3.2 The three-dimensional case

A parallel reasoning for the 3-dimensional Manhattan street network $M_3 = M(N_1,N_2,N_3)$, which will be generalized in the following subsection, leads to

$$\begin{pmatrix}
0 & \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 & 0\\
\lambda_1 & 0 & \frac{1}{\lambda_3} & \frac{1}{\lambda_2} & 0 & 0 & 0 & 0\\
\lambda_2 & \frac{1}{\lambda_3} & 0 & \frac{1}{\lambda_1} & 0 & 0 & 0 & 0\\
\lambda_3 & \frac{1}{\lambda_2} & \frac{1}{\lambda_1} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_3} & \frac{1}{\lambda_2} & \frac{1}{\lambda_1}\\
0 & 0 & 0 & 0 & \frac{1}{\lambda_3} & 0 & \lambda_1 & \lambda_2\\
0 & 0 & 0 & 0 & \frac{1}{\lambda_2} & \lambda_1 & 0 & \lambda_3\\
0 & 0 & 0 & 0 & \frac{1}{\lambda_1} & \lambda_2 & \lambda_3 & 0
\end{pmatrix}
\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\\ \epsilon\\ \zeta\\ \mu\\ \tau\end{pmatrix}
=\lambda
\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\\ \epsilon\\ \zeta\\ \mu\\ \tau\end{pmatrix}, \tag{18}$$

provided that $\lambda_1,\lambda_2,\lambda_3$ are eigenvalues of the directed cycles $C_{N_1}, C_{N_2}, C_{N_3}$, respectively. Thus, we can simplify the study of the eigenvalues of expression (18) to the computation of the zeros of the characteristic polynomial of the matrix

$$\begin{pmatrix}
0 & \lambda_1 & \lambda_2 & \lambda_3\\
\lambda_1 & 0 & \frac{1}{\lambda_3} & \frac{1}{\lambda_2}\\
\lambda_2 & \frac{1}{\lambda_3} & 0 & \frac{1}{\lambda_1}\\
\lambda_3 & \frac{1}{\lambda_2} & \frac{1}{\lambda_1} & 0
\end{pmatrix}, \tag{19}$$

which is

$$p_3(x) = -x^4 + Ax^2 + 2Bx + 3,$$

where

$$\begin{aligned}
A &:= \lambda_1^2+\lambda_2^2+\lambda_3^2+\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2},\\
B &:= \lambda_1\,\lambda_2\,\lambda_3+\frac{\lambda_1}{\lambda_2\,\lambda_3}+\frac{\lambda_2}{\lambda_1\,\lambda_3}+\frac{\lambda_3}{\lambda_1\,\lambda_2}.
\end{aligned}$$

Then, as 1 and -1 are always eigenvalues of the N-cycle C_N, there are two fixed sets of possible values of the coefficients A, B, namely $A = 6, B = 4$ and $A = 6, B = -4$. In the first case, the eigenvalues of the Manhattan street network are $\lambda = 3, -1$ and their opposites $\lambda = -3, 1$ correspond to the second assumption, according to Proposition 3.1.

Other possible situations that may help in the knowledge of the spectrum are the following:

- $\lambda_1 = \lambda_2 = 1$: $p(x) = (x + \lambda_3)(x + \frac{1}{\lambda_3})(x^2 - (\frac{1}{\lambda_3^2} + 1)x - 3)$.
- $\lambda_1 = 1, \lambda_2 = -1$: $p(x) = (x - \lambda_3)(x - \frac{1}{\lambda_3})(x^2 + (\frac{1}{\lambda_3^2} + 1)x - 3)$.
- If $N_1 \equiv 0 \pmod 4$, then $\lambda_1 = \pm i$:
  - $\lambda_1 = \pm i, \lambda_2 = \lambda_3 = \pm 1$: $p(x) = -(x^2 + 1)(x^2 - 3)$.
  - $\lambda_1 = \pm i, \lambda_2 = \pm 1$: $p(x) = -x^4 + (\lambda_3^2 + \lambda_3^{-2})x^2 + 3$.
- If $N_1 \equiv N_2 \equiv 0 \pmod 4$, then $\lambda_1 = \pm i, \lambda_2 = \pm i$:
  - $\lambda_1 = \pm i, \lambda_2 = \pm i, \lambda_3 = \pm 1$: $p(x) = -(x + 1)(x - 1)(x^2 + 3)$.

3.3 The general case revisited

To derive the spectrum of an n-dimensional Manhattan street network of dimension $n > 3$, it is useful to consider a new family of graphs, which are closely related to hypercubes and folded hypercubes. Accordingly, we begin by recalling the definition of the later.

The *folded* $(n+1)$*-cube*, usually denoted by $\tilde{Q}_n$, is obtained from the hypercube Q_n by adding an edge between each pair of antipodal vertices, that is, $x_1x_2\ldots x_n \sim \overline{x}_1\overline{x}_2\ldots\overline{x}_n$. Hence, $\tilde{Q}_n$ is an $(n+1)$-regular graph on 2^n vertices, with diameter $D = \lfloor (n+1)/2 \rfloor$ and distinct eigenvalues

$$\operatorname{ev} \tilde{Q}_n = \{n + 1 > n - 3 > n - 7 > \cdots\}$$

(see for instance [5]). An alternative way to get $\tilde{Q}_n$ is by identifying the antipodal vertices of the hypercube Q_{n+1}.

The graph that we call the *conjugate* n*-cube*, denoted $\overline{Q}_n$, has the same vertex set as the hipercube, $\mathbb{Z}_2^n$, and two vertices are adjacent whenever the corresponding strings differ in all components but one:

$$x_1x_2\ldots x_i\ldots x_n \quad \sim \quad \overline{x}_1\overline{x}_2\ldots x_i\ldots\overline{x}_n \quad (1 \le i \le n).$$

Thus, $\overline{Q}_n$ is an n-regular graph which, in fact, coincides either with the hypercube or the folded hypercube, depending on the dimension, as the following result shows.

Lemma 3.4. *Let $\overline{Q}_n$ be the conjugate hypercube defined above. Then,*

(a) $\overline{Q}_n \cong Q_n$ *if n is even.*

(b) $\overline{Q}_n \cong \tilde{Q}_{n-1} \cup \tilde{Q}_{n-1}$ *if n is odd.*

Proof. (*a*) When n is even, we can obtain $\overline{Q}_n$ from the hypercube Q_n by simply conjugating all vertices with odd Hamming weight. Notice that such vertices constitute one of the stable sets in Q_n (when seen as a bipartite graph), so that every vertex is either as it was or it becomes conjugated. Then, it is clear that the graph obtained has the same adjacencies as $\overline{Q}_n$.

(*b*) Assuming now that n is odd, we first consider the folded n-cube $\tilde{Q}_{n-1}$ obtained by identifying the antipodal vertices of the hypercube Q_n (as commented above). Hence,

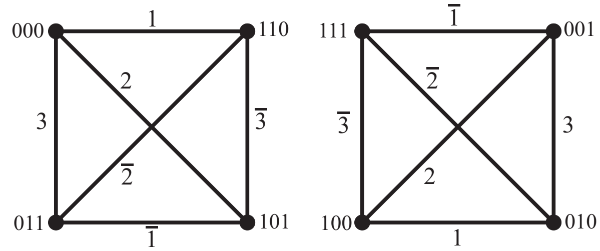


Figure 5: The conjugate hypercube $\overline{Q}_3$.

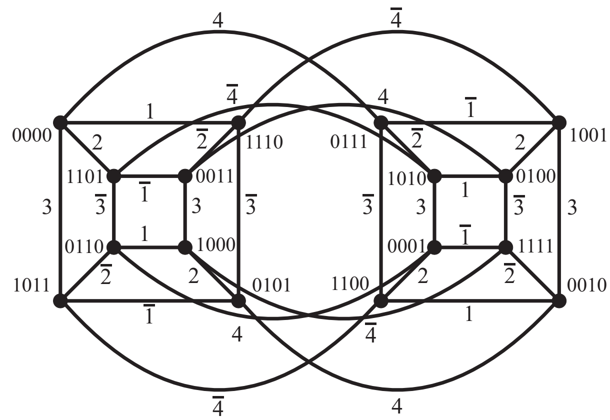


Figure 6: The conjugate hypercube $\overline{Q}_4$.

the vertices of $\tilde{Q}_{n-1}$ can be identified with the 2-sets $\{x_1x_2\ldots x_n,\ \overline{x}_1\overline{x}_2\ldots\overline{x}_n\}$, and the adjacencies are:

$$\{x_1x_2\ldots x_i\ldots x_n,\ \overline{x}_1\overline{x}_2\ldots\overline{x}_i\ldots\overline{x}_n\} \quad\sim\quad \{x_1x_2\ldots\overline{x_i}\ldots x_n,\ \overline{x}_1\overline{x}_2\ldots x_i\ldots\overline{x}_n\}.$$

But this, in turn, corresponds to the following two adjacencies in $\overline{Q}_n$ (one in each copy of $\tilde{Q}_{n-1}$)

$$\begin{aligned} x_1x_2\ldots x_{i-1}x_ix_{i+1}\ldots x_n &\sim \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-1}x_i\overline{x}_{i+1}\ldots\overline{x}_n,\\ \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-1}\overline{x}_i\overline{x}_{i+1}\ldots\overline{x}_n &\sim x_1x_2\ldots x_{i-1}\overline{x_i}x_{i+1}\ldots x_n. \end{aligned}$$

Notice that, in this case, the 2^n vertices of $\overline{Q}_n$ are partitioned into two sets (without edges between them), which are the vertex sets of the two copies of $\tilde{Q}_{n-1}$. Since n is odd, both sets have the same size, each of them containing all the sequences whose number of 1's has the same (even or odd) parity. This completes the proof. $\square$

By way of example, Figs. 5 and 6 show two particular cases of $\overline{Q}_n$ $(n = 3, 4)$ corresponding to the above cases (b) and (a), respectively.

The next result reduces the computation of the eigenvalues of the n-dimensional Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$ to the study of the spectra of the $n \times n$ matrix $\boldsymbol{W}$, corresponding to the adjacency matrix of a *weighted* conjugate n-cube, defined in the following way: let $i_0i_1\ldots i_{n-1}$ and $j_0j_1\ldots j_{n-1}$ be the binary expressions of the vertices i, j (row and column of $\boldsymbol{W}$). Then, the entries of $\boldsymbol{W} = (w_{ij})$ are

$$w_{ij} = \begin{cases} \lambda_k\,, & \text{if } i_m \neq j_m \text{ for all } m \neq k \text{ and } i_k = j_k = 0,\\ \lambda_k^{-1} = \overline{\lambda_k}\,, & \text{if } i_m \neq j_m \text{ for all } m \neq k \text{ and } i_k = j_k = 1,\\ 0\,, & \text{otherwise.} \end{cases} \tag{20}$$

where $i, j \in \{0, 1, \ldots, n-1\}$ and λ_k is an eigenvalue of the cycle C_{N_k}, $k = 1, 2, \ldots, n$.

Theorem 3.5. *For any given n integers $N_1, N_2, \ldots, N_n$, the different eigenvalues of the n-dimensional Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$ coincide with the distinct eigenvalues of the weighted conjugate n-cube $\overline{Q}_n^*$:*

$$\operatorname{ev} M_n = \operatorname{ev} \overline{Q}_n^*. \tag{21}$$

Proof. Let $\boldsymbol{u}^j = (u_0^j, u_1^j, \ldots, u_{N_j-1}^j)$ be an eigenvector of the cycle C_{N_j}, with corresponding eigenvalue λ_j. Thus, as stated before, its components satisfy $u_{k+1}^j = \lambda_j\, u_k^j$ and $u_{-k-1}^j = \frac{1}{\lambda_j}\, u_{-k}^j$. We assume that an eigenvector $\boldsymbol{w}$ of M_n, with unknown eigenvalue λ, has components (charges),

$$\begin{aligned} w_{(i_1,i_2,\ldots,i_n)} &= \alpha_{00\ldots0} u_{i_1}^1 u_{i_2}^2 \ldots u_{i_n}^n + \cdots + \alpha_{11\ldots1} u_{-i_1}^1 u_{-i_2}^2 \ldots u_{-i_n}^n \\ &= \sum_{x\in\mathbb{Z}_2^n} \alpha_{x_1x_2\ldots x_n}\, u^1_{(-1)^{x_1}\, i_1} u^2_{(-1)^{x_2}\, i_2} \ldots u^n_{(-1)^{x_n}\, i_n}, \end{aligned} \tag{22}$$

where $x = x_1x_2\ldots x_n \in \mathbb{Z}_2^n$. Then, as vertex $(i_1, i_2, \ldots, i_j, \ldots, i_n)$ in M_n is adjacent to vertices $(-i_1, -i_2, \ldots, i_j+1, \ldots, -i_n)$, $j = 1, 2, \ldots, n$, and using a reasoning parallel to that in Proposition 3.2, we obtain the following:

$$\begin{aligned} \lambda w_{(i_1,\ldots,i_n)} &= \sum_{x\in\mathbb{Z}_2^n} \lambda\alpha_x\, u^1_{(-1)^{x_1}\, i_1} u^2_{(-1)^{x_2}\, i_2} \ldots u^n_{(-1)^{x_n}\, i_n} \\ &= \sum_{j=1}^{n} \sum_{y\in\mathbb{Z}_2^n} \alpha_y\, u^1_{((-1)^{y_1+1})i_1} u^2_{((-1)^{y_2+1})i_2} \ldots u^j_{((-1)^{y_j})(i_j+1)} \ldots u^n_{((-1)^{y_n+1})i_n} \\ &= \sum_{y\in\mathbb{Z}_2^n} \sum_{j=1}^{n} \alpha_y u^1_{((-1)^{y_1+1})i_1} u^2_{((-1)^{y_2+1})i_2} \ldots \lambda_j^{(-1)^{y_j}}\, u^j_{((-1)^{y_j})i_j} \ldots u^n_{((-1)^{y_n+1})i_n} \\ &= \sum_{x\in\mathbb{Z}_2^n} \sum_{j=1}^{n} \alpha_{\overline{x}_1\overline{x}_2\ldots x_j\ldots\overline{x}_n} u^1_{((-1)^{\overline{x}_1+1})i_1} u^2_{((-1)^{\overline{x}_2+1})i_2} \ldots \lambda_j^{(-1)^{x_j}}\, u^j_{((-1)^{x_j})i_j} \ldots u^n_{((-1)^{\overline{x}_n+1})i_n} \\ &= \sum_{x\in\mathbb{Z}_2^n} \sum_{j=1}^{n} \lambda_j^{(-1)^{x_j}}\, \alpha_{\overline{x}_1\overline{x}_2\ldots x_j\ldots\overline{x}_n} u^1_{(-1)^{x_1} i_1} u^2_{(-1)^{x_2} i_2} \ldots u^j_{(-1)^{x_j} i_j} \ldots u^n_{(-1)^{x_n} i_n}. \end{aligned}$$

Thus, since the above equalities are satisfied for any $u_{i_1}^1, u_{i_2}^2, \ldots, u_{i_n}^n$, the next matrix equation must hold:

$$\boldsymbol{W} \begin{pmatrix} \alpha_{00\ldots0} \\ \alpha_{0\ldots01} \\ \vdots \\ \alpha_{11\ldots1} \end{pmatrix} = \lambda \begin{pmatrix} \alpha_{00\ldots0} \\ \alpha_{0\ldots01} \\ \vdots \\ \alpha_{11\ldots1} \end{pmatrix}. \tag{23}$$

Consequently, λ is an eigenvalue of the weighted conjugate n-cube $\overline{Q}_n^*$, as claimed. Notice that, in particular, when $\lambda_i = 1$, $i = 1, \ldots, n$, the matrix $\boldsymbol{W}$ is the adjacency matrix of Q_n and we retrieve the eigenvalues of the n-cube. □

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